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LINEAR ALGEBRA AND ITS APPLICATIONS

Matrix cyclization over complex polynomials

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Abstract

Let A be an $n \times n, B$ an $n \times m$ complex polynomial matrix. The following open problem is investigated: does there exist a complex polynomial $(m \times n)$-matrix F such that $A + BF$ has a cyclic vector in the image of B? An explicit solution of the problem is given for the following generic situation: $n \geqslant 4, [B, AB, \dots, A^{n-1}B]$ is rightinvertible (necessary), the entries of a specific normalized form of (A, B) do *not* satisfy certain polynomial equations, and one specific entry has at least one simple zero outside a certain finite set. The exceptional equations are given explicitely and can easily be checked for. © 1998 Elsevier Science Inc. All rights reserved.

1. Introduction

Let $\mathbb{C}[y]$ be the ring of complex polynomials in one indeterminate $y, A \in \mathbb{C}[y]^{n \times n}, B \in \mathbb{C}[y]^{n \times m}$. The problem we will investigate is the following: does there exist $F \in \mathbb{C}[y]^{m \times n}$ and $u \in \mathbb{C}[y]^{m \times 1}$, s.t. the matrix $A + BF$ has Bu as a cyclic vector, i.e. s.t.

$$[Bu, (A + BF)Bu, \dots, (A + BF)^{n-1}Bu] \quad \text{is invertible.} \tag{1}$$

This problem is called the matrix(-feedback)-cyclization problem (FC) for the pair (A, B). Its origin is in control theory. See e.g. [1,2]. Obviously the so-called *reachability* of (A, B), i.e. the rightinvertibility of the *reachability matrix* $[B, AB, \dots, A^{n-1}B]$, is a necessary condition for the solvability of FC for

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(A, B). In fact a very special kind of right inversion (see below) is required to solve the problem. A 15-year-old conjecture (see [3]) says that for any reachable pair (A, B) the FC-problem has a solution.

Despite the seemingly elementary character of the problem, the attempts to understand and solve the problem till now indicate that one cannot expect an easy solution (see e.g. the comments in [4], p. 282). An interesting information in this context is that for *real* polynomial matrix pairs FC is in general not solvable, whereas for reachable matrix pairs whose entries are analytic functions FC is solvable [5]. Some more details on the background of the FC-problem are given, e.g. in [1], see also the numerous notes in [2].

In [5] an equivalent symmetric formulation of the FC-problem is given which simplifies considerably its treatment via direct solution of the underlying matrix/determinantal equations. In Section 2 we therefore briefly recall this version of FC. In Section 3 we show, how one can generalize the ideas for $n = 4$ from [5] to obtain a detailed solution for reachable pairs (A, B) of arbitrary dimension n except for a very special nongeneric type of reachable pairs which are described explicitly.

In Section 4 proofs are given for the series of lemmata which make the solution in Section 3 possible.

Some concluding remarks will follow in the Final Section 5.

In the sequel it will tacitly be assumed that $n \geqslant 4$. For $n \leqslant 3$ the FC-problem is solvable for reachable pairs, see [3,6].

2. A reformulation of the FC-problem

Our starting point will be the following result from [5,9].

Theorem 1. *Let* $n \geqslant 4$.

(a) FC is solvable for any n-dimensional reachable pair and any $m \geqslant 1$. $\Longleftrightarrow$ *FC is solvable for any n- dimensional reachable pair with* $m = 2$.

(b) FC is solvable for any n-dimensional reachable pair with $m = 2$. $\Longleftrightarrow$ *For all complex polynomials* $a, c_1, \ldots, c_{n-2}, b$, *where* $\gcd(a, b) = 1$, *one can find complex polynomials* $v_1, \ldots, v_{n-2}, u$ *and* $Q \in \mathbb{C}[y]^{2\times 2}$ *with* $\det Q = 1$, *such that with*

$$A = \begin{bmatrix} 0 & & & & 0 \\ 1 & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & 1 & \ddots & \vdots \\ 0 & & & a & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & c_1 \\ \vdots & \vdots \\ \cdot & c_{n-2} \\ 0 & b \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & & & & 0 \\ 1 & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & 1 & \ddots & \vdots \\ 0 & & & 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & v_1 \\ \vdots & \vdots \\ \cdot & v_{n-2} \\ 0 & u \end{bmatrix},$$

and with the reachability matrices

$$\mathscr{R} = [B, AB, \ldots, A^{n-1}B], \quad \mathscr{S} = [D, CD, \ldots, C^{n-1}D]$$

and with $\mathscr{Q} = \operatorname{diag}(Q, \ldots, Q)$ *one has*

$$\det(\mathscr{R}\mathscr{Q}^{t}\mathscr{S}) = 1 \tag{2}$$

or explicitly:

$$\det\left(\begin{bmatrix} 1 & 0 & 0 & 0 & & 0 & 0 \\ 0 & c_1 & 1 & 0 & & \cdot & \cdot \\ \cdot & \cdot & 0 & c_1 & \cdots & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & c_{n-2} & \cdot & \cdot & & \cdot & \cdot \\ 0 & b & 0 & ac_{n-2} & & a & 0 \end{bmatrix} \operatorname{diag}(Q, \ldots, Q) \begin{bmatrix} 1 & 0 & \cdot & \cdots & 0 & 0 \\ 0 & v_1 & \cdot & \cdots & v_{n-2} & u \\ 0 & 1 & 0 & \cdots & \cdot & 0 \\ 0 & 0 & v_1 & \cdots & \cdot & v_{n-2} \\ & & \vdots & & & \\ 0 & \cdot & \cdot & \cdots & \cdot & 1 \\ 0 & \cdot & \cdot & \cdots & \cdot & 0 \end{bmatrix}\right) = 1.$$

The last and rather long statement in other words just says: consider only pairs of the simple structure A, B and find an equally structured (but with $a = 1$!) pair C, D s.t. up to a base change $B \to BQ$ (or $D \to D^{t}Q$) the reachability matrix $\mathscr{S}$ becomes a rightinverse of $\mathscr{R}$. Since $\gcd(a, b) = 1$ the pair (A, B) is reachable and thus $\mathscr{R}$ is rightinvertible. The theorem says that the solvability of FC is equivalent to the existence of a rightinverse of the rather special form $\mathscr{Q}^{t}\mathscr{S}$.

Although later on we will try to solve directly Eq. (2), it will be helpful to recall, what this equation means in terms of the control process behind it. For this let

$$q^{(1)} = \begin{bmatrix} q_{11} \\ q_{21} \end{bmatrix} \text{ and } \quad q^{(2)} = \begin{bmatrix} q_{12} \\ q_{22} \end{bmatrix} \text{ be the columns of } Q.$$

Then the matrix in Eq. (2) can be written as follows:

$$\begin{aligned}\mathscr{R}\mathscr{Q}^{t}\mathscr{S} = [&Bq^{(1)}, ABq^{(1)} + v_1 Bq^{(2)}, \ldots, A^{n-1}Bq^{(1)} \\ &+ v_1 A^{(n-2)}Bq^{(2)} + \cdots + v_{n-2}ABq^{(2)} + uBq^{(2)}]. \end{aligned} \tag{3}$$

The columns of this matrix are the states which can be reached by the control process

$$x_0 = 0, \quad x_{k+1} = Ax_k + Bw_k \quad \text{for } k \geqslant 0 \tag{4a}$$

by applying the sequence of control vectors

$$w_0 = q^{(1)}, \quad w_1 = v_1 q^{(2)}, \ldots, w_{n-2} = v_{n-2}q^{(2)}, \quad w_{n-1} = uq^{(2)}. \tag{4b}$$

Note that the control vector $q^{(1)}$ is used only once at the beginning. After that only multiples of $q^{(2)}$ occur. If these multiples can be chosen in such a way that $x_1, \ldots, x_n$ form a basis of $\mathbb{C}[y]^n$, then FC is solvable and vice versa.

In what follows we will always tacitly assume *a and b to be nonzero or even nonconstant* if necessary. If a or b are zero, then b or a must be a nonzero constant, since $\gcd(a, b) = 1$. If a is a nonzero constant, then FC is already solved, for then the first column of B is a cyclic vector for A. If b is a nonzero constant, then a solution for FC is obtained by setting $v_1 = \cdots = v_{n-2} = 0$, $u = b^{-1}(1 - a)$, and Q the identity matrix.

We will also rely on the well-known fact from control theory that the solvability of the FC-problem is not affected when transforming the pair (A, B) by transformations of the type

$$(A, B) \to (S(A + BF)S^{-1}, SBT) =: (\tilde{A}, \tilde{B}),$$

where $S \in GL_n(\mathbb{C}[y]), T \in GL_2(\mathbb{C}[y])$ which form the so-called feedback group. Of course, we only can admit those transformations which yield a pair $(\tilde{A}, \tilde{B})$ of exactly the same structure as (A, B) which means $1 - s$ and $0 - s$ as prescribed in A and B. These transformations form a subgroup of the feedback group which we denote by FCG_n. In Section 4, proof of Lemma 8, an example for the action of this group will appear.

3. Towards a solution for $n \geqslant 4$

Note that – to improve readability – all proofs of technical lemmata are postponed to Section 4. Let $M := \mathscr{R}\mathscr{Q}\mathscr{S}$, where $\mathscr{R}, \mathscr{Q}, \mathscr{S}$ are as in Theorem 1, and let $Q = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}$. Despite the simple structure of these matrices, the determinant of M as a polynomial in $a, c_1, \dots, c_{n-2}, b, v_1, \dots, v_{n-2} u$ and $q_{11}, q_{12}, q_{21}, q_{22}$ has a rapidly growing number of terms (e.g. 42, 215, 1111, 6366 for $n = 4, 5, 6, 7$).

Since the pair (A, B) is "nearly" cyclic, i.e. the first column of B is "almost" a cyclic vector for A, it makes sense to try to "intervene" as late as possible in terms of the underlying control process (4) in order to correct the "bad" influence of a and at the same time to minimize the number of free parameters. This philosophy leads to the assumption:

$$v_1 = \cdots = v_{n-3} := 0. \tag{5}$$

Then $v_{n-2} := v, u, q_{11}, q_{12}, q_{21}, q_{22}$ are the remaining parameters to be chosen appropriately subject to the condition $\det Q = 1$.

It is relatively easy to see that setting also $v_{n-2} = 0$, then the intervention by u in general comes too late.

Lemma 2. *Suppose (5). Then*

$$\det M = a q_{11}^n + \det Q(d_u u + d_v v) + \det Q^2 d_{vv} v^2, \tag{6}$$

where

$$\begin{aligned} d_u &= (-1)^{n-2} \det [B, ABq^{(1)}, \dots, A^{n-2} Bq^{(1)}], \\ d_v &= 2(-1)^{n-3} \det [B, ABq^{(1)}, \dots, A^{n-3} Bq^{(1)}, A^{n-1} Bq^{(1)}], \\ d_{vv} &= -\det [B, AB] \quad \text{if } n = 4, \\ d_{vv} &= -\det [B, AB, A^2 Bq^{(1)}, \dots, A^{n-3} Bq^{(1)}] \quad \textit{for } n \geqslant 5. \end{aligned}$$

Moreover $d_u, d_{vv} d_{vv}$ are homogeneous as polynomials in q_{11}, q_{21}. The degrees are: $n-2$ for d_u if $b \neq 0$, $n-2$ for d_v if a and c_{n-2} are nonzero and $n-4$ for d_{vv} if b and c_{n-3} are nonzero.

Note that as a consequence of the symmetrized version of the FC-problem the entries q_{12}, q_{22} of the "control direction" $q^{(2)}$ do *not* occur in d_u, d_v, d_{vv}. This allows us to proceed as follows, when trying to solve the equation $\det M = 1$: choose q_{11}, q_{21} subject to the condition $\gcd(q_{11}, q_{21}) = 1$, complete somehow to a matrix Q with $\det Q = 1$, choose at the same time v and u, all this, of course, in order to get $\det M = 1$.

Proceeding in this way we can forget about q_{12}, q_{22} and $\det Q$ in (6) and the equation to be solved is "merely"

$$a q_{11}^n + d_u u + d_v v + d_{vv} d_{vv} v^2 = 1 \tag{7}$$

subject to $\gcd(q_{11}, q_{21}) = 1$.

Examples

$$n = 4: d_u = bq_{11}^2 - 2ac_1c_2q_{11}q_{21} + ac_1^3q_{21}^2,$$
$$d_v = 2aq_{11}(c_2q_{11} - c_1q_{21}),$$
$$d_{vv} = ac_2^2 - bc_1.$$
$$n = 5: d_u = bq_{11}^3 - a(2c_1c_3 + c_2^2)q_{11}^2q_{21} + 3ac_1^2c_2q_{11}q_{21}^2 - ac_1^4q_{21}^3,$$
$$d_v = 2aq_{11}(c_3q_{11}^2 - 2c_1c_2q_{11}q_{21} + c_1^3q_{21}^2),$$
$$d_{vv} = (c_3^2a - c_2b)q_{11} + (c_1^2b - 2ac_1c_2c_3 + ac_2^3)q_{21}.$$

3.1. Solution strategy for Eq. (7)

Considering d_u, d_v, d_{vv} as polynomials in q_{11}, q_{21} over $\mathbb{C}[y]$ we can define g to be the gcd of all occuring coefficients. Let $d_u = g\bar{d}_u, d_v = g\bar{d}_v, d_{vv} = g\bar{d}_{vv}$.
Instead of solving (7) we can solve at first

$$aq_{11}^n + g\lambda = 1 \tag{8}$$

for q_{11}. Choose any q_{21} s.t. $\gcd(q_{11}, q_{21}) = 1$. We will take advantage of this freedom below. After inserting the solution q_{11} together with q_{21} into $\bar{d}_u, \bar{d}_v, \bar{d}_{vv}$ solve

$$\bar{d}_u u + \bar{d}_v v + \bar{d}_{vv} v^2 = \lambda. \tag{9}$$

Clearly (8), (9) have a solution if and only if (7) has a solution. Solvability and solutions of (8) are described by Lemmata 3 and 4.

Lemma 3. *(a) Eq. (8) has a solution* $\Longleftrightarrow \gcd(a, g) = 1$. *(b) Let* $\Lambda = \{q \in \mathbb{C}[y] : aq^n \equiv 1 \bmod g\}, q \in \Lambda$ *and* $p \in \mathbb{C}[y]$. *Let also* ζ *be the* primitive nth root of unity from $\mathbb{C}$. *Then*: $p \in \Lambda \Longleftrightarrow \exists g_0, \ldots, g_{n-1} \in \mathbb{C}[y]$:
(i) $g = g_0 \cdots g_{n-1}$;
(ii) $\gcd(g_i, g_j) = 1$ *for* $0 \leqslant i, j \leqslant n - 1$ *with* $i \neq j$;
(iii) $p \equiv \zeta^i q \bmod g_i$ *for* $0 \leqslant i \leqslant n - 1$.

Lemma 4. *One always has* $\gcd(a, g) = 1$.

As a consequence, if we have chosen a solution q_{11} for (8), then there remains still some freedom to modify q_{11}. E.g. for any $u \in \mathbb{C}[y]$ the polynomial $q_{11} + \mu g$ is still a solution. Choose $g_0 = g, g_i = 1$ for $i > 0$ in Lemma 3. Any solution is necessarily coprime with g.

Coming back to our study of (7) we realize that "only" Eq. (9) remains to be solved subject to $\gcd(q_{11}, q_{21}) = 1$, where q_{11} is already fixed mod$g$. Since u occurs linearly in (9) one can apply a standard modular technique: solve *locally* at eventually multiple zeros of $\bar{d}_u$ (see Section 3.2) and then find a global solution by *interpolation* (Chinese Remainder Theorem). The latter step always works. Of course, we can only proceed along these lines if $\bar{d}_u$ is not the zero polynomial.

Lemma 5. *The solution* q_{11} *of (8) can be assumed to be coprime with any given nonzero polynomial and if* $\gcd(q_{11}, a) = 1$, *then* $\bar{d}_u$ *is not the zero polynomial independently of the choice of* q_{21}.

3.2. Local solution for Eq. (9)

At first we expose a basic relation between d_u, d_v, d_{vv}.

Lemma 6. *Without any restrictions one has*

$$d_v^2 - 4aq_{11}^n d_{vv} \equiv 0 \bmod d_u. \tag{10}$$

This relation – which actually is a rather natural determinantal identity as the proof in Section 4 reveals – shows us that common zeros of d_u and d_{vv} must be zeros of d_v and then (9) might not have a solution.

It is therefore worthwhile to understand under which circumstances d_{vv} can be even identically zero.

Lemma 7. d_{vv} *is the zero polynomial as a polynomial in* q_{11}, q_{21} *over* $\mathbb{C}[y]$ *iff*

$$b^j_{C_{n-2-j}} = a^j c^{j+1}_{n-2} \quad \textit{for} \quad q \leqslant j \leqslant n-3.$$

This shows that triviality of d_{vv} is a rather "rare" event. Our solution method for the FC-problem requires the following *first* of two assumptions:

(A1) $\quad d_{vv} \neq 0$

which will be assumed from now on. But even if $d_{vv} \neq 0$, then $\bar{d}_u$ and $\bar{d}_{vv}$ might have common zeros (once chosen q_{11} and q_{21}). We will show now, how to avoid such common zeros.

Let Z be the subfield of $\mathbb{C}$ generated by the coefficients of $a, c_1, \ldots, c_{n-2}, b$ and q_{11} and choose $q_{21} := t \in \mathbb{C}$ to be transcendental over Z. Note that by this choice one has $\gcd(q_{11}, q_{21}) = 1$ independently of q_{11}.

The choice $q_{21} = t$ has furthermore the consequence that *the zeros of $\bar{d}_u$ are all transcendental over Z*. For if the zero y_0 of $\bar{d}_u$ were algebraic over Z, then $\bar{d}_u(y_0) \in Z(y_0)[t]$. However, $\bar{d}_u$ is neither the zero polynomial nor its coefficients as a polynomial in t have a common zero (remember homogenicity from Lemma 2).

Let now y_0 be such a transcendental zero of $\bar{d}_u$, then t is algebraic over the rational function field $Z(y_0)$ and we can look of its minimal equation.

Lemma 8. *If a has a simple prime divisor not dividing c_1, then d_u is a minimal polynomial for t. If $n = 4$, then such a prime divisor can always be provided modulo the action of the group FCG_n (given $d_{vv} \neq 0$).*

Therefore our *second* assumption will be from now on, that:

(A2) *a has a simple prime divisor not dividing c_1.*

Now we get $d_{vv}(y_0) \neq 0$ as an immediate consequence of Lemma 8, since d_{vv} is of lower degree in q_{21} (see Lemma 2).

Only now we are ready to solve (9) at zeros y_0 of d_u, where still q_{11} belongs to a solution of (8) and $q_{21} = t$ as above. Our Eq. (9) becomes then merely

$$\bar{d}_v(y_0)v + \bar{d}_{vv}(y_0)v^2 = \lambda(y_0) \tag{11}$$

and always has one or two solutions. Fortunately one never has only one solution! This can be seen by looking at the discriminant which is

$$\mathrm{dis}(a_0) := \bar{d}_v^2(y_0) + 4\bar{d}_{vv}(y_0)\lambda(y_0).$$

But by (8) one has $1 - a(y_0)q_{11}(y_0)^n = g(y_0)\lambda(y_0)$ and therefore by (10): $g(y_0)^2\mathrm{dis}(y_0) + 4d_{vv} = 0$. Since $d_{vv} \neq 0$ we must have $\mathrm{dis}(y_0) \neq 0$. The simplicity of the roots of (11) has the great advantage that a Newton–Hensel lifting of these solutions is possible in case y_0 is not a simple zero of $\bar{d}_u$. Altogether this means that we have achieved a local solution of (9) subject only to the condition $d_{vv} \neq 0$.

Remember that after having done this we can always get a global solution of (9) by interpolation. Once given a solution of (9) together with the solution q_{11}, λ of (8) we obtain a solution of (7), i.e. a solution of the FC-problem in all cases, where the two conditions (A1) and (A2) are met.

4. Proofs of lemmata in Section 3

Proof of Lemma 2. When (5) is assumed, then the determinant to be determined is

$$\det M = [M', A^{n-2}Bq^{(1)} + vBq^{(2)}, A^{n-1}Bq^{(1)} + vABq^{(2)} + uBq^{(2)}],$$

where $M' = [Bq^{(1)}, ABq^{(1)}, \ldots, A^{n-3}Bq^{(1)}]$. Expansion by columns gives

$$\det M = \det M_{11} + v\det M_{1v} + u\det M_{1u} + v\det M_{v1} + v^2\det M_{vv} + vu\det M_{vu},$$

where $M_{11} = [M', A^{n-2}Bq^{(1)}, A^{n-1}Bq^{(1)}], M_{1v} = [M', A^{n-2}Bq^{(1)}, ABq^{(2)}]$, $M_{1u} = [M', A^{n-1}Bq^{(1)}, Bq^{(2)}]$, $M_{v1} = [M', Bq^{(2)}, A^{n-1}Bq^{(1)}]$, $M_{vv} = [M', Bq^{(2)}, ABq^{(2)}]$, $M_{vu} = [M', Bq^{(2)}, Bq^{(2)}]$. Clearly $\det M_{11} = aq_{11}^n$ and $\det M_{vu} = 0$. Less obviously: $\det M_{1v} = \det M_{v1}$. This can be seen by rewriting the determinants as follows:

$$\det M_{1v} = (-1)^{n-3}\det[Bq^{(1)}, AU],$$
$$\det M_{v1} = (-1)^{n-3}\det[U, A^{n-1}Bq^{(1)}],$$

where $U = [BQ, ABq^{(1)}, \ldots, A^{n-3}Bq^{(1)}]$. Let $u_{(1)}, \ldots, u_{(n)}$ be the rows of U, then

$$\det M_{1v} = (-1)^{n-3}\det\begin{bmatrix} q_{11} & 0 \\ * & u_1 \\ & \vdots \\ & u_{n-2} \\ * & au_{n-1} \end{bmatrix}$$
$$= (-1)^{n-3}\det\begin{bmatrix} u_1 & 0 \\ \vdots & \vdots \\ u_{n-1} & 0 \\ u_{11} & aq_{11} \end{bmatrix} = \det M_{v1}$$

and furthermore $2\det M_{v1} = (\det Q)d_v$ (d_v as stated in Lemma 2) by the Cauchy–Binet Formula.

Also by Cauchy–Binet one obtains $\det M_{vv} = (\det Q)^2 d_{vv}$ and $\det M_{1u} = (\det Q)d_u$. Homogeneity of d_u, d_v, d_{uv} is obvious. As for the degrees one observes easily that $bq_{11}^{n-2}, ac_{n-2}q_{11}^{n-2}, bc_{n-3}q_{11}^{n-4}$ are the respective leading terms with respect to q_{11}. □

Proof of Lemma 3. (a) Has been proved in [6].
(b) If $p, q \in \Lambda$, then $a(p^n - q^n) \equiv 0 \bmod g$. Since by (a) one knows that $\gcd(a, g) = 1$, the latter means $p^n - q^n \equiv 0 \bmod g$ which in turn is equivalent to: $\prod_{i=0}^{n-1}(p - \zeta^i q) \equiv 0 \bmod g$. Now (i)–(iii) and the converse are easily deduced. □

Proof of Lemma 4. At the end of the proof of Lemma 2 it was observed that bq_{11}^{n-2} is one of the terms of d_u. Thus $g|b$ by definition of g. $\gcd(a, b) = 1$ has been assumed from the beginning. □

Proof of Lemma 5. Let p be a nonzero polynomial and q_{11} be a solution of (8). We apply an old trick from ring theory: Let μ be the product of all normed prime polynomials which divide p but not q_{11} and let $z \in \mathbb{C}\setminus\{0\}$, then $\gcd(q_{11} + z\mu g, p) = 1$. According to the remarks directly after Lemma 3, $q_{11} + z\mu g$ is also a solution of (8).
Let now $\gcd(q_{11}, a) = 1$. Remembering the formula for d_u from Lemma 2 one calculates $d_u = bq_{11}^{n-2} \bmod a$. Since also $\gcd(b, a) = 1$, d_u and $\bar{d}_u$ cannot be the zero polynomial whatever q_{21} might be. □

Proof of Lemma 6. We will use the notations $M', M_{1v}, M_{1u}, M_{vv}$ from Lemma 2. Form the matrix $MM = \begin{bmatrix} M_{1v} & N \\ L & M_{v1} \end{bmatrix}$ where all matrices are $n \times n$ and where N is the zero matrix except for the (n,n)−position, which is aq_{11}, as in M_{v1} and where L is the zero matrix except for the last column, which is $ABq^{(2)}$ as in M_{1v}.

By elementary row and column operations MM can be transformed to

$$MMM = \begin{bmatrix} M_{1u} & 0 \\ M^{**} & M^{*} \end{bmatrix},$$

where $M^* = [M', ABq^{(2)}, A^{n-1}Bq^{(1)}]$ and such that $\det MMM = -\det MM; M^{**}$ need not be specified. Now

$$\det MM = \det M_{1v} \det M_{v1} - (-1)^{11} aq_{11}^{n} \det M_{vv}$$

and

$$\det MMM = \det M_{1u} \cdot \det M^{*}.$$

Together with the relations from the end of the proof of Lemma 2 and the observation that $\det Q$ divides $\det M^*$ this gives

$$d_v^2 - 4aq_{11}^n d_{vv} \equiv 0 \bmod d_u. \quad \square$$

Proof of Lemma 7. Lemma 7 serves to describe the exceptional pairs where $d_{vv} = 0$ and has no influence on the further steps of the solution procedure. For reasons of space we therefore only prove here the following weaker statement $(j = 1)$: $d_{vv} = 0 \Rightarrow bc_{n-3} = ac_{n-2}^2$.

If $n = 4$, then $d_{vv} = bc_1 - ac_2^2$. Let therefore $n \geqslant 5$. In Lemma 2 we saw

$$d_{vv} = \det \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & c_1 & 1 & 0 & 0 & \dots & 0 \\ \cdot & \cdot & 0 & c_1 & q_{11} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & q_{11} \\ 0 & c_{n-2} & 0 & c_{n-3} & * & \cdots & * \\ 0 & b & 0 & ac_{n-2} & * & \cdots & * \end{bmatrix}.$$

Therefore the leading term in q_{11} of d_{vv} is: $\pm(ac_{n-2}^2 - bc_{n-3})q_{11}^{n-4}$. $\square$

Proof of Lemma 8. Consider $d_u(y_0)$ as a polynomial in $Z(y_0)[t]$. By the representation for d_u from Lemma 2 we obtain

$$d_u = bq_{11}^{n-2} + a(\dots) + ac_1^{n-1}t_t^{n-2}.$$

If therefore a contains a simple prime divisor which does not divide c_1, then d_u is a Einstein polynomial and thus irreducible. We will now show how one can provide such a prime factor of a, when $n = 4$ and $d_{vv} \neq 0$, as is assumed any-way. Let $f \in \mathbb{C}[y]$ and let "$\sim$" denote "action of the full feedback group". One easily verifies:

$$\left(\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & c_1 \\ 0 & c_2 \\ 0 & b \end{bmatrix}\right) \sim \left(\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & c_1 f & 0 \\ 0 & 1 & c_2 f & 0 \\ 0 & 0 & a+bf & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & c_1 \\ 0 & c_2 \\ 0 & b \end{bmatrix}\right)$$

$$\sim \left(\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a+bf & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & \tilde{c}_1 \\ 0 & c_2 \\ 0 & b \end{bmatrix}\right),$$

where $\tilde{c}_1 = c_2^2 f + c_1$. Thus the first and the third pair are equivalent under the action of the subgroup FCG_n (see Section 1). We have $d_{vv} = ac_2^2 - bc_1$ for both the first *and* the third pair. Let $\deg f > \deg d_{vv}$, $\gcd(f, a) = 1$ by the lemma in [7] we can modify f by a nonzero constant such that $a + bf$ becomes square free, what will be assumed now. Suppose $a + bf | \tilde{c}_1$. Then for any zero y_0 of $a + bf$ we have $b(y_0) \neq 0$ since $\gcd(a, b) = 1$ and therefore

$$f(y_0) = -\frac{a(y_0)}{b(y_0)} \quad \text{and} \quad b(y_0)\tilde{c}_1(y_0)$$
$$= b(y_0)c_1(y_0) - a(y_0)c_2(y_0)^2 = d_{vv}(y_0) = 0.$$

Since y_0 is an arbitrary zero of $a + bf$, a degree argument yields: $d_{vv} = 0$. Thus there must be a (simple!) zero of $a + bf$ which leaves $\tilde{c}_1$ nonzero. □

5. Concluding remarks

The solution strategy developed in [5] for $n = 4$ and $d_{vv} \neq 0$ turned out to be generalizable for arbitrary n after some modifications and one additional assumption for $n > 4$. A solution could be given (Sections 3 and 4) for the generic subproblem of the FC-problem where $d_{vv} \neq 0$ (assumption (A1)) and where a has at least one simple zero *not* in common with c_1 (assumption (A2)). The latter condition is redundant for $n = 4$ and possibly for all n. The first condition on d_{vv} can *not* be avoided. If $d_{vv} = 0$, then one can show that this true for feedback equivalent pairs also. Therefore this case must be treated separately. The condition $d_{vv} = 0$ can be checked by calculating the determinant given in Lemma 2. The condition on the zeros of a requires gcd-calculations.

By generic we mean: for all $\delta > 0$ the set of exceptional matrix pairs with degrees of $a, c_1, \ldots, c_{n-2}, b$ bounded by δ are contained in a proper algebraic subvariety in the finite-dimensional space of all admissible matrix pairs. Note that the property $\gcd(a, b) = 1$ is generic. Lemma 7 indicates that the exceptional surface $d_{vv} = 0$ is rather peculiar. For the exceptional pairs a solution of the FC-problem still has to be found. The conjecture in [3] says that there is one.

The solution in Section 4 works for pairs of the rather special form given in Theorem 1. When starting with an arbitrary reachable pair, there is an algorithmic procedure developed in [7] and programmed in [8] which does the transformations to the normalized form. A Maple-package can be obtained from the author.

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